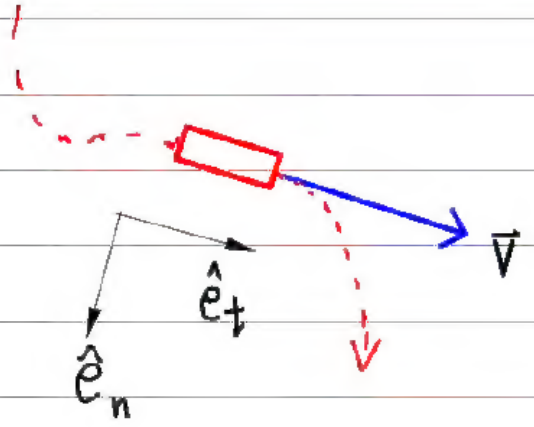
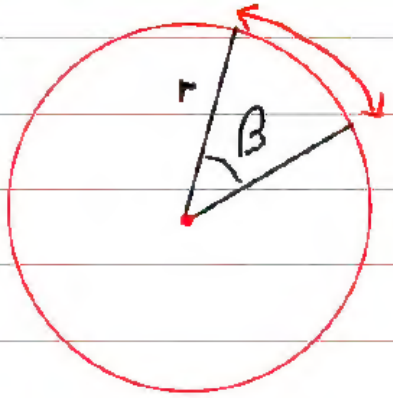
Meriam 2/5 : n-t coordinates



We would like to find expressions for the following in n-t coordinates:

$$\vec{r}, \vec{v}, \vec{a}$$

To help us, review definition of **radian measurement**:



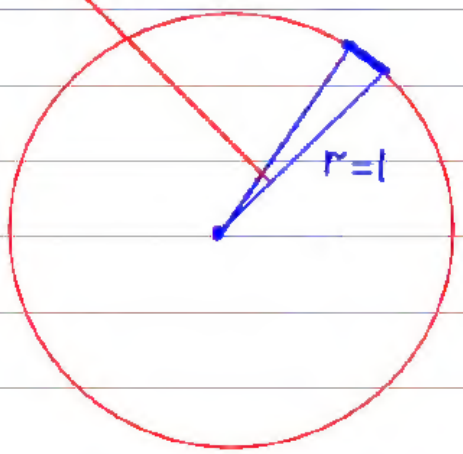
$\beta = 45^\circ$ (degrees)

or $\beta = \dfrac{\text{arc length}}{\text{radius}}$ (radians)

- radians are dimensionless
- $360^\circ \sim \dfrac{2\pi r}{r}$ radians $= 2\pi$
- For our drawing above,
  $\beta = 45^\circ \sim \dfrac{2\pi}{8} = \pi/4$ radians

Very useful geometric property

small angle $\Delta\theta$



Isosceles triangle:

two sides of length 1, & one very short side

What is the length of the short side of the triangle in the limit that $\Delta\theta$ becomes very, very small ?

Answer: In the limit of very small $\Delta\theta$, the line becomes identical to the red chord length that lies underneath

but $\Delta\theta = \dfrac{\text{chord length}}{\text{radius}}$

→ For an isosceles triangle with small angle $\Delta\theta$ (radians) and two sides of length 1,

Length of third side = $\Delta\theta$

Objective: Find expressions for $\vec{r}$, $\vec{V}$, and $\vec{a}$ in the n–t coordinate system

Recall:

1/ We have no origin in the n–t coordinate system

2/ $\hat{e}_t$ is in the direction of motion, changes with time.
$\hat{e}_n$ is always perpendicular (normal) to $\hat{e}_t$

3/ $\hat{e}_n$ is pointed towards inside of curved motion
(undefined for straight-line motion)

Solution:

(a) $\vec{r}$ is a vector going from the origin to the position of point P at time t.
But we have no origin in the n–t system.
→ position $\vec{r}$ is undefined

(b) $\vec{V}$ has magnitude v, and points in direction $\hat{e}_t$:
→ $\vec{V} = \frac{d\vec{r}}{dt} = v\hat{e}_t$

(c) $\vec{a} = \frac{d\vec{V}}{dt}$: Relationship between $\vec{a}$ and $\vec{V}$ is the primary use of n–t system

$$\vec{a} = a_n \hat{e}_n + a_t \hat{e}_t$$

I need to find easy-to-use expressions for a_n & a_t

Use product rule for differentiation:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\hat{e}_t)}{dt} = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \quad ★$$

Why is eqn ★ so complicated?

Contrast it to equation for $\vec{a}$ in x-y coordinates:

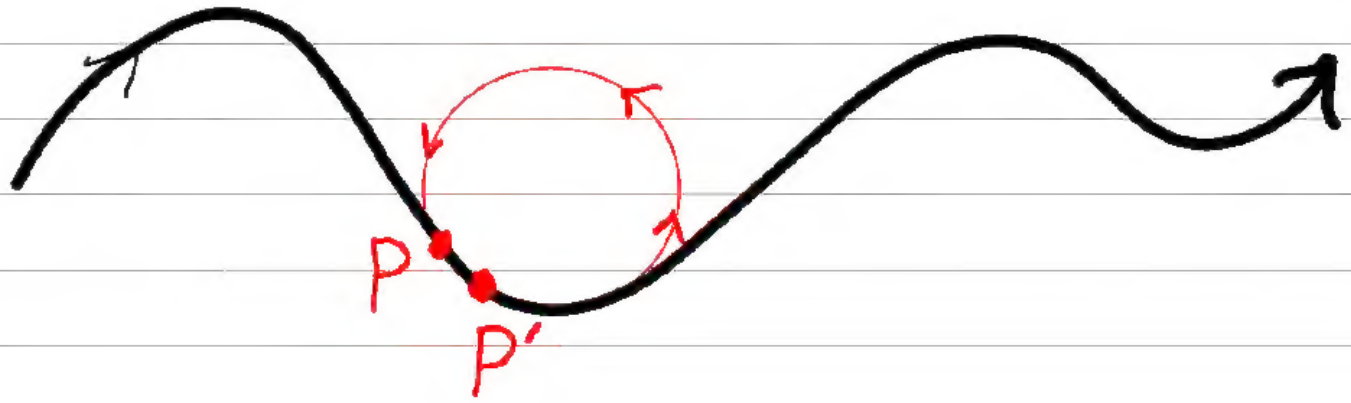
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\dot{x}\hat{i} + \dot{y}\hat{j})}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$$

Goal: Get a simplified expression for $\frac{d\hat{e}_t}{dt}$ → ★

Note: If trajectory is straight,

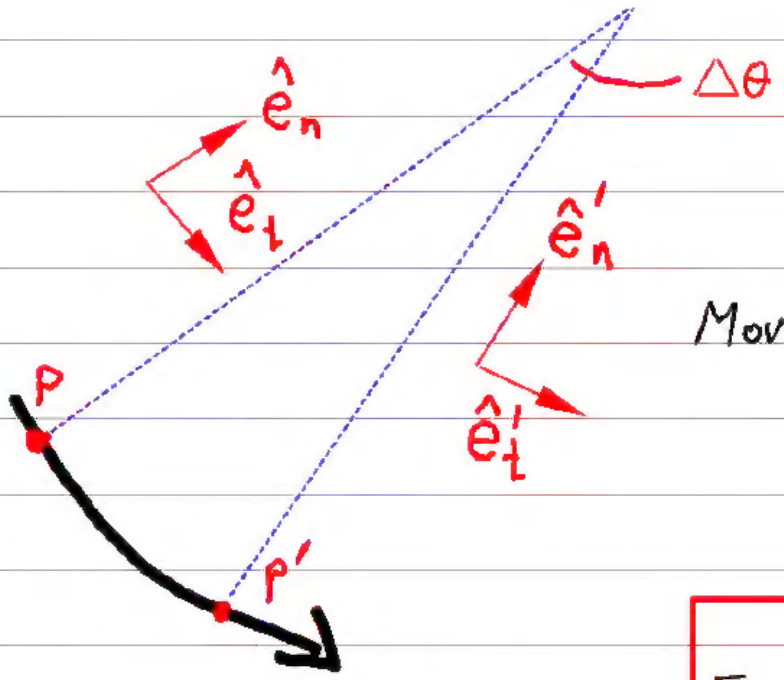
then $\frac{d\hat{e}_t}{dt} = 0$, → problem is rectilinear motion

otherwise, trajectory is curved



Particle is at point P, and then moves on to point P' in a very short time period Δt.

Let's magnify the region around points P and P':

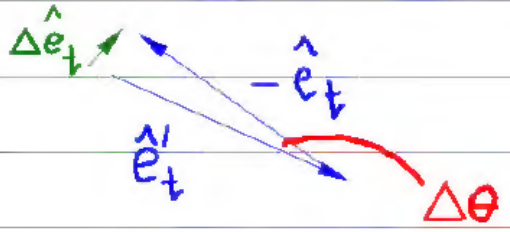


Movement of P in time Δt

Find $\dfrac{d\hat{e}_t}{dt} = \lim_{\Delta t \to 0}\left\{\dfrac{\Delta \hat{e}_t}{\Delta t}\right\}$

where: $\Delta \hat{e}_t = \hat{e}_t' - \hat{e}_t$; $\Delta t = t' - t$

or $\dfrac{\Delta \hat{e}_t}{\Delta t} = \dfrac{\hat{e}_t'}{\Delta t} - \dfrac{\hat{e}_t}{\Delta t}$



$\rightarrow$ $\dfrac{\Delta \hat{e}_t}{\Delta t} = \dfrac{\Delta\theta\ \hat{e}_n}{\Delta t}$

Take limit as $\Delta t \to 0$ $\rightarrow$ $\dfrac{d\hat{e}_t}{dt} = \dot{\theta}\hat{e}_n$

Repeating Eqn ★ page 4(d), $\vec{a} = \dot{v}\,\hat{e}_t + v\frac{d\hat{e}_t}{dt}$

$$\rightarrow \quad \vec{a} = \dot{v}\,\hat{e}_t + v\dot{\theta}\,\hat{e}_n$$

$$= a_t\,\hat{e}_t + a_n\,\hat{e}_n$$

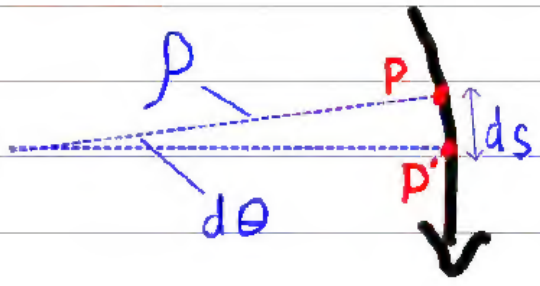
where $\dot{\theta}$ = radians/s $= s^{-1}$

& magnitude of acceleration :

$$|\vec{a}| = \sqrt{a_t^2 + a_n^2\,(+a_z^2)}$$

$a_t = \dot{v}$; $a_n = v\dot{\theta}$

Consider a car turning,
Radius of curvature $= \rho$



$d\theta = ds/\rho \rightarrow ds = \rho\,d\theta$

$$\rightarrow v = |\vec{v}| = \frac{ds}{dt} = \frac{\rho\,d\theta}{dt} = \rho\dot{\theta}$$

$$\rightarrow \dot{\theta} = v/\rho \rightarrow a_n = v\dot{\theta} = v\frac{v}{\rho} = \frac{v^2}{\rho}$$

$$\rightarrow \quad \vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n$$

Summary

- n-t coordinates: $\hat{e}_n$, $\hat{e}_t$
- Unit vectors depend on instantaneous velocity of P → They vary with time
- $\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$

Next Lecture

- Practical examples for n-t coordinates
- Read 2/6: polar (r–θ) coordinates